# Engineering Notes

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## Comments on the Lawrence Equation for Low-Aspect-Ratio Wings

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### I. Introduction

RECENTLY, the authors had an occasion to compute the chordwise lift distribution of a wing in an incompressible flow. The method suggested by Lawrence[1] was used and programmed on a personal computer. Computations showed that some of the computed results presented by Lawrence are in error, although all his derivations are correct. During the investigation, an improved method of calculating the singular integral appearing in the Lawrence equation was also developed. In this Note, we present an improved method of solving the Lawrence equation, compare the results with those of Lawrence, and point out the erroneous computed results of Lawrence.

### II. Lawrence Equation

The Lawrence equation is[1]

$$k(x) = (1/2)g(x) + (1/4)\int_{-1}^{1} g'(\xi)\,[1+K(x,\xi)]\,d\xi \tag{1}$$

where

$$K(x,\xi) = [(x-\xi)^2 + s^2(x)]^{1/2}/(x-\xi) \tag{2a}$$

$$k(x) = \int_{-s(x)}^{s(x)} w(x,y)\,[s^2(x)-y^2]^{1/2}\,dy = s^2(x)\int_{-1}^{1} w(x,Y)(1-Y^2)^{1/2}\,dy, \quad Y = y/s(x) \tag{2b}$$

$$g'(x) = \frac{dg}{dx} = \int_{-s(x)}^{s(x)} u(x,y)\,dy \tag{3}$$

where the Cartesian coordinates $(x,y)$ and the local wing semispan function are as shown in Fig. 1. The local wing span is of length $2s(x)$ and the wing root chord of length 2. The $s(x)$ must be nondecreasing along the positive $x$ direction and the wing trailing edge must be normal to the freestream. The $u$ is the parametric lift per unit area, $w$ the negative of the local slope of airfoil in the stream direction, and $g'(x)$ the parametric lift per unit chord. Equation (1) refers to the approximation when the wing and its wake are represented by an appropriate vortex sheet lying on the $xy$ plane $(z=0)$.

The problem of interest is the following: Given $w(x,y)$, find $g'(x)$. Equation (1) is an integro-differential equation with a singular kernal. The integral must be interpreted in a Cauchy principal value sense.

### III. Lawrence's Method of Solution

Lawrence chose to represent $g(x)$ in the following form:

$$g(\vartheta) = (\pi-\vartheta)(A_0+A_1) + \sum_{r=1}^{N-1} (A_{r-1}-A_{r+1})(\sin r\vartheta)/r \tag{4}$$

where $A_0, A_1, \ldots, A_{N-2}$ are unknown constants, $A_{N-1} = A_N = 0$, and $\vartheta = \arccos(x)$. Consequently,

$$\frac{dg}{dx} = A_0 \tan(\vartheta/2) + 2\sum_{r=1}^{N-2} A_r \sin r\vartheta \tag{5}$$

and the lift coefficient $C_L$, pitching-moment coefficient $C_M$ about the wing-root chord leading edge, and the location $(X_{ac}, 0)$ of the center of pressure are, respectively,

$$C_L = (4\pi/S)(A_0+A_1) \tag{6a}$$

$$C_M = -(2\pi/SC_r)(A_0+A_2) \tag{6b}$$

$$X_{ac}/C_r = 0.5 - 0.25(A_0-A_2)/(A_0+A_1) \tag{6c}$$

where $S$ is the wing planform area and $C_r = 2$ the wing-root-chord length.

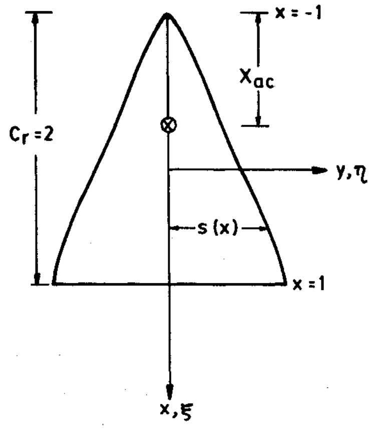


Fig. 1 Wing nomenclature and coordinate system.

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After substituting for $g(x)$ in Eq. (1), we obtain

$$k(\vartheta) = (\pi/4)[F_i(\vartheta) - F_0(\vartheta)]A_0 + (\pi/4)\sum_{r=1}^{N-2} \times [F_{r+1}(\vartheta) - F_{r-1}(\vartheta)]A_r = \sum_{r=0}^{N-2} A_r D_r(x) \quad (7)$$

where the definition of $D_r(x)$ is obvious, and

$$F_0(\vartheta) = 2\vartheta/\pi + H_o(\vartheta, s) - 3 \quad (8a)$$

$$F_r(\vartheta) = (2/\pi r)\sin r\vartheta + s(\vartheta)\sin r\vartheta/\sin\vartheta + H_r(\vartheta, s), \qquad r \geq 1 \quad (8b)$$

$$H_r(\vartheta, s) = (1/\pi)\int_0^\pi \cos r\varphi \, H(\vartheta, s, \varphi)\, d\varphi \quad (9)$$

$$H(\vartheta, s, \varphi) = \{[(\cos\varphi - \cos\vartheta)^2 + s^2(x)]^{1/2} - s(\vartheta)\}/(\cos\varphi - \cos\vartheta) \quad (10)$$

where $\varphi = \arccos(\xi)$, $N \geq 4$.

The $(N-1)$ unknown constants $A_0, A_1, \ldots, A_{N-2}$ are obtained by satisfying Eq. (7) at the $(N-1)$ collocation points given by $\vartheta_i = i\pi/N$, $i = 1, 2, \ldots, N-1$. The quantity $H_r(\vartheta, s)$ is calculated using the trapezoidal rule on $\varphi$, leading to

$$H_{r_i}(s) = H_r(\vartheta_i, s) = \left(\frac{1}{2M}\right)\left[H(\vartheta_i, s_i, 0) + (-1)^r H(\vartheta_i, s_i, \pi) + 2\sum_{m=1}^{M-1}\cos(mr\pi/M) \times H(\vartheta_i, s_i, m\pi/M)\right] s_i = s(\vartheta_i) \quad (11)$$

where $M(>N)$ is a suitably chosen positive integer, but such that $\vartheta \neq m\pi/M$ for any possible value of $i$ or $m$ because by Eq. (10), $H$ will then be singular. This is insured if $M$ is not an integer multiple of $N$.

## IV. Improved Method of Solution

The improvement suggested is finding a better way of numerically calculating the integral in Eq. (1). We note that Eq. (5) may also be expressed as

$$\frac{dg}{dx} = \left[A_0(1-x) + 2(1-x^2)^{1/2} + \sum_{r=1}^{N-2} A_r \sin r\vartheta\right] \Big/ (1-x^2)^{1/2} \quad (12)$$

so that Eq. (1) becomes

$$k(x) = \sum_{r=0}^{N-2} A_r[S_r(x) + T_r(x)] = \sum_{r=0}^{N-2} A_r D_r(x) \quad (13)$$

where $D_r(x) = S_r(x) + T_r(x)$. Further,

$$S_0(x) = \pi - \vartheta + \sin\vartheta, \quad S_1(x) = \pi - \vartheta + (1/2)\sin 2\vartheta$$

$$S_r(x) = [\sin(r+1)\vartheta]/(r+1) - [\sin(r-1)\vartheta]/(r-1) \qquad r \geq 2 \quad (14)$$

$$T_0(x) = \int_{-1}^{1} (1-\xi^2)^{-1/2}[1 + K(x,\xi)](1-\xi)\, d\xi$$

$$T_r(x) = 2\int_{-1}^{1} (1-\xi^2)^{-1/2}[1 + K(x,\xi)] \times (1-\xi^2)^{1/2} \sin r\varphi \, d\xi, \qquad r \geq 1 \quad (15)$$

The foregoing integrals have the generic form

$$I(x) = \int_{-1}^{1} (1-\xi^2)^{-1/2} f(x,\xi)(x-\xi)^1 \, d\xi$$

where $f(x,\xi)$, $-1 \leq x$, $\xi \leq 1$, is a well-behaved function. For such integrals, Stark[2] has given an elegant quadrature formula

$$I(x_j) = \sum_{i=1}^{N} e_i f(x_j, \xi_i)(x_j - \xi_i)^{-1}, \quad j = 1, 2, \ldots, N-1 \quad (16)$$

$$x_j = \cos(j\pi/N), \qquad \xi_i = \cos[(2i-1)\pi/2N], \quad e_i = \pi/N \quad (17)$$

where $I(x_j)$ is exactly evaluated if $f(x,\xi)$ is a polynomial of degree $\leq 2N$ in $\xi$. For comparable accuracy and on the basis of numerical experimentation, the calculation of $D_r$ using the Stark quadrature rule to compute $T_r$ was estimated to be an order of magnitude faster than Lawrence's method. The accuracy of the present method was established by the evidence that Lawrence's method approached the values calculated by the present method with increasing $M$ for rectangular wings.

For a well-behaved $w(x,y)$, $k(x)$ may be evaluated by the following well-known Gaussian quadrature formula for the weight function $(1-Y^2)^{1/2}$

$$k(x_j) = \left(\frac{\pi}{N+1}\right) s^2(x_j) \times \sum_{i=1}^{N}(1-Y_i^2)\, w[x_j, s(x_j)Y_i]$$

$$Y_i = \cos[i\pi/(N+1)], \qquad i = 1, 2, \ldots, N \quad (18)$$

**Table 1 d $C_L$/d$\alpha$ and $X_{ac}/C_r$ as a function of aspect ratio for rectangular wing and delta wing[a]**

| Aspect ratio values: | | 0.25 | 0.50 | 1.00 | 2.00 | 4.00 |
|---|---|---|---|---|---|---|
| | | | Rectangular wing | | | |
| $dC_L/d\alpha$ | Lawrence | 0.3922 | 0.7738 | 1.4585 | 2.4684 | 3.6140 |
| | Bera and Suresh | 0.3922 | 0.7738 | 1.4585 | 2.4684 | 3.6140 |
| $X_{ac}/C_r$ | Lawrence | 0.0637 | 0.1113 | 0.1673 | 0.2089 | 0.2308 |
| | Bera and Suresh | 0.0637 | 0.1113 | 0.1673 | 0.2089 | 0.2308 |
| | | | Delta wing | | | |
| $dC_L/d\alpha$ | Lawrence | 0.3676 | 0.6971 | 1.2653 | 2.1565 | 3.3859 |
| | Bera and Suresh | 0.3675 | 0.6971 | 1.2653 | 2.1565 | 3.3860 |
| $X_{ac}/C_r$ | Lawrence | 0.6456 | 0.6300 | 0.6032 | 0.5639 | 0.5141 |
| | Bera and Suresh | 0.6451 | 0.6295 | 0.6030 | 0.5638 | 0.5141 |

[a]Calculations were made with $N = 8$ and $M = 61$.

For a chosen $N$ and a suitably chosen high enough value of $M$, the results produced by Lawrence's method and the present method are indistinguishable.

## V. Numerical Results

Table 1 lists the values of $dC_L/d\alpha$ and $X_{ac}/C_r$, where $\alpha$ is the wing incidence for various aspect ratios of rectangular and delta wings, as computed from Lawrence's method and the present method. The agreement is excellent. Although not tabulated here, similar excellent agreement exists for $dg/dx$ in all cases.

However, when a check on the results plotted in Lawrence's paper was made, it was found that his $dg/dx$ plots were in gross error for both rectangular and delta wings, which are clearly computational errors.

## VI. Conclusions

A faster, accurate algorithm has been devised to solve the Lawrence equation for low-aspect-ratio wings. Errors of numerical computation occurring in Lawrence's original paper are noted.

## References

[1] Lawrence, H. R., "The Lift Distribution on Low Aspect Ratio Wings at Subsonic Speeds," *Journal of the Aeronautical Sciences,* Vol. 18, Oct. 1951, pp. 683-695.

[2] Stark, V. J. E., "A Generalized Quadrature Formula for Cauchy Integrals," *AIAA Journal,* Vol. 9, Sept. 1971, pp. 1854-1855.

# Experimental and Analytical Analysis of Grid Fin Configurations

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## Introduction

AN unusual concept in missile fin design has led to a theoretical analysis and a series of subsonic wind-tunnel tests on grid fin configurations. The fin design, appropriately termed a "grid" fin, is an all-moveable fin that consists of (usually) high-aspect-ratio members of constant chord arranged in a grid-work pattern. The fin outer perimeter may be square, rectangular, octagonal, or virtually any shape consisting of straight members connected end to end, while the inner section of the fin may consist of any arbitrary grid pattern. The internal fin density may range anywhere from an open cavity to a dense honeycomb arrangement. Obviously, because of this flexibility, drag tailoring is easily obtainable. Additionally, these designs prove advantageous where control surfaces are span limited or situations where small actuator requirements are a high priority. Fins are normally mounted transverse to the longitudinal axis of the missile so that flow passes through the grid members. The angle of attack of the fin is achieved by a simple rotation of the fin about its horizontal axis in much the same way a conventional fin control surface is deflected.

The theoretical aerodynamic analysis of these grid fins logically lends itself to a vortex-lattice formulation. Primary coefficients of interest are normal force, root-chord bending moment, hinge moments, and drag. Vortex-lattice methods have been applied to complex configurations such as multiplanes, nonplanar wings, and entire aircraft lifting surfaces with intricate geometries, resulting in modern computerized methods such as those developed by Feifel[1] and Margason and Lamar.[2]

## Theory

Standard vortex-lattice theory is centered around the Biot–Savart law and may be viewed as a constant-pressure paneling method suited to thin lifting surfaces of moderate to high aspect ratios. During the current investigation, a compressible vortex-lattice formulation was developed and embodied in a revision of the Biot–Savart law. In the analysis, the fin configuration is divided into a network of subpanels spaced uniformly (or nonuniformly) on each fin member in the spanwise and chordwise direction. A lifting vortex of unknown strength is located along the quarter-chord line of each subpanel as prescribed originally by Pistolesi[3] as the optimum location for spanwise vortices and later validated by Byrd[4] for chordwise vortices. A pair of trailing vortices is shed along the panel edges parallel to the uniform flow, downstream to infinity and forms a so-called horseshoe vortex.

The boundary condition of no flow through the surface is satisfied at the three-quarter-chord lateral center of each subpanel and, thus, completes Pistolesi's (1/4–3/4) theorem. Imposition of the boundary condition results in a system of equations that can be solved to yield the strengths of the vortices. Once the vortex strengths are known, the fin-sectional aerodynamic characteristics may be determined by appropriate summations of the chordwise direction. Finally, the total fin aerodynamic characteristics are computed by standard numerical integration techniques in the spanwise direction.

Vortex-lattice equations may be derived directly from the Biot–Savart equation, which defines the velocity induced at a field point by a vortex filament. In vector form, the linearized compressible form of this equation is

$$\bar{V}(x,y,z) = \frac{-\Gamma\beta^2}{4\pi}\int \frac{\bar{r}\times\overline{dl}}{|\bar{R}_\beta|^3} \tag{1}$$

where $\Gamma$ is the filament strength, and $\bar{r}$ is the vector from a field point $(x,y,z)$ to a differential length of the vortex filament $\overline{dl}$. Compressibility enters the equation through a typical Mach number term in the numerator $(\beta^2 = 1.0 - M_\infty^2)$ and through the elliptic radius defined by

$$|\bar{R}_\beta| = \{(x-x_o)^2 + \beta^2[(y-y_o)^2 + (z-z_o)^2]\}^{0.5} \tag{2}$$

The boundary conditions of no flow through the fin at control points can be written in matrix notation as

$$[A]\,(\Gamma) = [B] \tag{3}$$

The $[A]$ matrix is known as the influence matrix and is composed of the downwash terms given by Eq. (1). The $[B]$ matrix is governed by the angle of attack and any other known upwash or downwash terms, and for the present case is simply

$$B_i = (\sin\alpha)_i, \qquad i = 1,\, nc \tag{4}$$

where $nc$ is the total number of grid-fin control points. Once the vortex strengths are found by matrix algebra, the aerodynamic characteristics of the lifting surface may be computed through utilization of the Kutta–Joukowsky theorem for the

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